

Scaling properties of first-passage quantities on the fractal and transfractal scale free networks

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Fractal (or transfractal) and scale free characters are common properties of real life network systems. It is of great significance to uncover the effect of these characters on the dynamic processes taking place on complex medias. In this paper, we consider the random walk process on a kind of fractal (or transfractal) scale free networks, which also called as (u, v) flowers, and we focus on the global first passage time (GFPT) and first return time (FRT). Here, we present method to derive exactly the probability generation function, mean and variance of the GFPT and FRT for a given hub (i.e., node with the highest degree) and then the scaling properties of the mean and the variance of the GFPT and FRT are disclosed. Our results show that, for the case of $u > 1$, while the networks are fractals, the mean of the GFPT scales with the volume of the network as $\langle GMFPT_t \rangle \sim N_t^{2/d_s}$, where $\langle * \rangle$ denotes the mean of random variable $*$, N_t is the volume of the network with generation t and d_s is the spectral dimension of the network; but, for the case of $u = 1$, while the networks are nonfractals, the mean of the GFPT scales as $\langle GMFPT_t \rangle \sim N_t^{2/\tilde{d}_s}$, where \tilde{d}_s is the transspectral dimension of the network, which is introduced in this paper. Results also show that, the variance of the GFPT scales as $Var(GFPT_t) \sim \langle GFPT_t \rangle^2$, where $Var(*)$ denotes variance of of random variable $*$; whereas the variance of the FRT scales as $Var(FRT_t) \sim \langle FRT_t \rangle \langle GFPT_t \rangle$. Our results imply that for the case that the networks are nonfractals, the mean and the variance of the GFPT are not controlled by the spectral dimension (i.e., $d_s = 2$), but they are controlled by the transspectral dimension. In order to evaluate the fluctuation of the GFPT and FRT, we also calculate the reduced moments of the the GFPT and FRT and find that, in the limit of large size, the reduced moment of the FRT tends to be infinite, whereas the reduced moment of the GFPT is almost a const. Therefore, on the (u, v) flowers of large size, the fluctuation of the FRT is huge, whereas the fluctuation of the GFPT is much smaller.

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I. INTRODUCTION

It is well known that many real-life networks display scale-free character¹ and this character has profound impact on dynamics taking place on scale-free networks²⁻⁵. Nevertheless, scale-free behavior cannot reflect all the structural information of real networks. It was acknowledged that several naturally occurring scale-free networks exhibit fractal or transfractal scaling^{6,7}. Taking into account fractal (or transfractal) scaling of scale-free networks can lead to a better understanding of how the underlying systems work⁷⁻⁹. However, It is difficult to determine exactly the effect of the fractal and scale-free characters on the dynamics on the real networks. In 2007, Rozenfeld *et al* introduced a kind of scale free networks which have fractal (or transfractal) character. These networks are controlled by two integral parameters (i.e., $1 \leq u \leq v$) and they are also called as (u, v) flowers. For $u > 1$, they are “large-world” and fractals. For $u = 1$, they are small-world and nonfractals, but they are referred as transfractals¹⁰. These networks are of great

importance because many dynamic properties can be exactly determined and then one can uncover the effect of the scale-free and fractal characters on the dynamical processes taking place on them by assigning u and v with different values¹¹⁻¹⁴. The pseudofractal scale-free web¹⁵, which has attracted lots of interest in the past several years¹⁶⁻²¹, is just an example of (u, v) flower with $u = 1$ and $v = 2$.

Here we focus on random walks, which is a fundamental dynamic process taking place on complex medias²²⁻²⁴, on the (u, v) flowers. The quantities we are interested in are the first passage time (FPT), which is the time it takes a random walker to reach a given site for the first time, and first return time (FRT), which is the time it takes a random walker to return to the starting site for the first time²⁵⁻³². In the past several years, a lot of work was devoted to analyze the mean^{12,33-42} and the variance^{25,43-46} of the two random variables on different networks. The mean provides valuable estimate of the random variable and the variance is good measure on whether the estimate provided by the mean is reliable. There are also some work which discloses the relation of between spectral dimension d_s and the mean and variance of FPT. For example, Haynes *et al* found that on some special fractal lattices, the mean of FPT scale with

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the volume of network as N^{2/d_s} and the variance of FPT scale with the volume of network as N^{4/d_s} ⁴⁶. Hwang *et al* found that the mean of FPT on scale free networks are affected by the spectral dimension d_s and the exponent of the degree distribution⁴⁷.

For random walks on the (u, v) flowers, the mean of FPT (MFPT) to a given hub (i.e. nodes with the highest degree) and the average of the MFPTs from all possible starting nodes are obtained for some special u and v ^{11,12}. The spectral dimension of the (u, v) flowers with shortcuts are founded¹³ and the relation between fractal dimension and the MFPT on special kind of (u, v) flowers is also analyzed¹⁴. However, the exactly results of the mean of FPT for any parameters u and v , the variance of the FPT and FRT, the relations among the mean and variance of FPT and FRT, the relation between the spectral dimension and the mean and variance of FPT and FRT, are still all unknown.

In this paper, we study the mean and variance of FPT and FRT on the general (u, v) flowers. Note that the FRT and FPT are deeply affected by the source or target site. We don't intend to enumerate all the possible cases and analyze them. On the contrary, we only consider the FRT for a given hub and the global FPT (GFPT) to a given hub, which is the average of the FPTs for arriving at a given hub from any possible starting site, with the probability that the random walker starts from node v is $d_v/(2E_t)$, where E_t is the total numbers of edges of the network and d_v is the degree of node v . Here we present method to calculate exactly the mean and the variance of the GFPT and FRT for a given hub and disclose the factors which affect them. Firstly, we analyze and obtain the recurrence relations of the probability generating functions (PGF) of the GFPT and FRT. Then, exploiting the probability generating function tool, we obtain the recurrence relations of the first and second moment of the GFPT and FRT. Finally, we obtain exactly the mean and variance of the GFPT and FRT by solving the recurrence relations. Results show that $\text{Var}(GFPT_t) \sim \langle GFPT_t \rangle^2$ and $\text{Var}(FRT_t) \sim \langle FRT_t \rangle \langle GFPT_t \rangle$, where $\text{Var}(GFPT_t)$ and $\text{Var}(FRT_t)$ denote the variance of the GFPT and the FRT, $\langle GFPT_t \rangle$ and $\langle FRT_t \rangle$ denote the mean of the GFPT and the FRT respectively. Results also show that, for the case of $u > 1$, while the networks are fractals, the mean and the variance of the GFPT scale with the volume of the networks, denoted by N_t , as $\langle GFPT_t \rangle \sim N_t^{2/d_s}$ and $\text{Var}(GFPT_t) \sim N_t^{4/d_s}$, which imply that both the mean and the variance of the GFPT are controlled by the spectral dimension d_s . However, for the case of $u = 1$, while the networks are nonfractals, the mean and the variance of the GFPT are not controlled by the spectral dimension $d_s = 2$, but they are controlled by the transspectral dimension \bar{d}_s and the mean and the variance of the GFPT scale with the volume of the networks as $\langle GFPT_t \rangle \sim N_t^{2/\bar{d}_s}$ and $\text{Var}(GFPT_t) \sim N_t^{4/\bar{d}_s}$.

In order to evaluate the fluctuation of the GFPT and FRT, we calculate the reduced moments⁴⁶ of the two random variables and find that, in the limit of large size (i.e.,

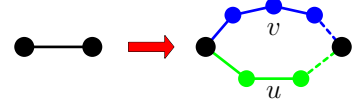


FIG. 1. Iterative construction method of the (u, v) flowers. The (u, v) flower of generation t ($t > 0$), denoted by $G(t)$, is obtained from $G(t-1)$ by replacing every edge of $G(t-1)$ by two parallel paths with lengths u and v ($1 \leq u \leq v$) on the right-hand side of the arrow.

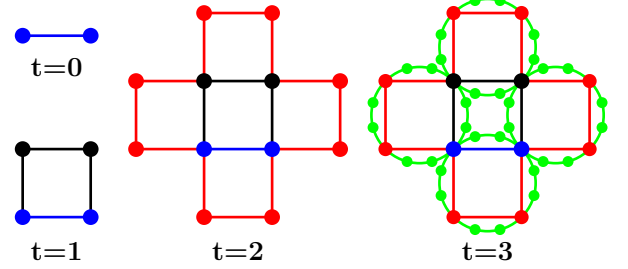


FIG. 2. The construction of the $(1, 3)$ flower with generation $t = 0, 1, 2, 3$.

while $N_t \rightarrow \infty$), the reduced moment of the FRT tends to be infinite, whereas the reduced moment of the GFPT is almost a const. Therefore, on the (u, v) flowers of large size, the FRT has huge fluctuation and the estimate provided by its mean is unreliable, whereas the fluctuation of the GFPT is much smaller and the estimate provided by its mean is more reliable.

II. NETWORK MODEL

The networks considered here, also called as (u, v) flowers ($1 \leq u \leq v$), are deterministically growing networks which can be constructed iteratively¹⁰. Let $G(t)$ denote the (u, v) flower of generation t ($t \geq 0$). The construction of the (u, v) flower starts from ($t = 0$) two nodes connected by an edge, which corresponds to $G(0)$. For $t \geq 1$, $G(t)$ is obtained by replacing every edges of $G(t-1)$ by two parallel paths of u and v edges long. Fig. 1 shows the iterative construction method of the (u, v) flowers and Fig. 2, Fig. 3 show the constructions of the $(1, 3)$ flower and the $(2, 2)$ flower with generation $t = 0, 1, 2, 3$. Let $w = u + v$. It is easy to see that the total number of edges for $G(t)$ is $E_t = w^t$ and the total number of nodes for $G(t)$ is $N_t = \frac{(w-2)w^t + w}{w-1}$ ^{10,14}.

The (u, v) flowers display rich behavior in their topological structure. They follow a power-law degree distribution $P(k) \sim k^{-\gamma}$ with the exponent $\gamma = 1 + \ln(u + v)/\ln 2$. For $u > 1$, the networks are “large-world” and fractals with the fractal dimension $d_f = \ln(u + v)/\ln u$ ¹⁰, the walk dimension $d_w = \ln(uv)/\ln u$ and the spectral dimension $d_s = 2\ln(u + v)/\ln(uv)$ ¹³. For $u = 1$, the

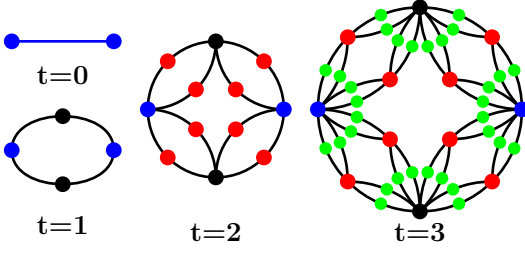


FIG. 3. The construction of the (2, 2) flower with generation $t = 0, 1, 2, 3$.

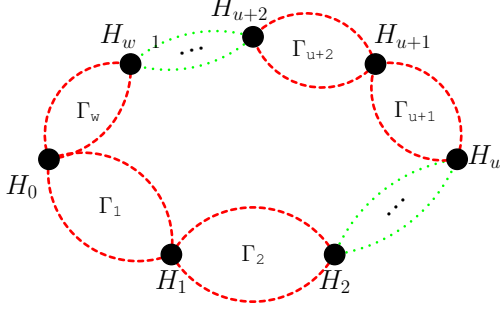


FIG. 4. Alternative construction of the (u, v) flower which highlights self-similarity: the network of generation t , denoted by $G(t)$, is composed of $w \equiv u + v$ copies, called subunits, of $G(t - 1)$ which are labeled as $\Gamma_1, \Gamma_2, \dots, \Gamma_w$, and connected to one another at its w hubs, denoted by H_0, H_1, \dots, H_{w-1} .

networks are small-world and nonfractals with spectral dimension $d_s = 2^{13}$; They have infinite fractal dimension, walk dimension and finite transfractal dimension $\tilde{d}_f = \ln(1 + v)/(v - 1)$ and transwalk dimension $\tilde{d}_w = \ln(v)/(v - 1)$. Therefore they are called as transfractals¹⁰. We can also similarly define their transspectral dimension by $\tilde{d}_s = 2\tilde{d}_f/\tilde{d}_w = 2\ln(1 + v)/\ln(v)$ ⁴⁸.

The network also has an equivalent construction method which highlights its self-similarity¹⁰. Referring to Fig. 4, in order to obtain $G(t)$, one can make w copies of $G(t - 1)$ and join them at their hubs (i.e., nodes with the highest degree) denoted by H_0, H_1, \dots, H_{w-1} . In such a way, $G(t)$ is composed of w copies of $G(t - 1)$ labeled as $\Gamma_1, \Gamma_2, \dots, \Gamma_w$, which are connected with each other by the w hubs.

III. METHOD OF EXACTLY CALCULATING THE PROBABILITY GENERATING FUNCTION OF FRT AND GFPT

In this section we analyze and present the recurrence relations which the probability generating functions of the FRT for a given hub and the GFPT to a given hub satisfy. Not loss generality, we only analyze the FRT for hub H_0 and GFPT to hub H_u .

Let $F_{i \rightarrow j}(t)$ be FPT from site i to site j on net-

work $G(t)$. Therefore, $F_{i \rightarrow i}(t)$ is just the first return time on network $G(t)$ for random walk starting from site i . Let $P\{F_{i \rightarrow j}(t) = n\}$ represents the probability that $F_{i \rightarrow j}(t) = n$. Thus, the probability generating functions of FPT from hub H_0 to hub H_u is given by

$$\Phi_{FPT}(t, z) = \sum_{n=0}^{+\infty} z^n P\{F_{H_0 \rightarrow H_u}(t) = n\}, \quad (1)$$

and the probability generating functions of FRT for hub H_0 is given by

$$\Phi_{FRT}(t, z) = \sum_{n=0}^{+\infty} z^n P\{F_{H_0 \rightarrow H_0}(t) = n\}. \quad (2)$$

In the steady state⁴², the probability of finding the random walker at node i is given by $d_i/2E_t$, where d_i is the degree of node i . Averaging $F_{i \rightarrow H_u}(t)$ over all starting node i , the GFPT to hub H_u is defined by

$$F_{H_u}(t) = \sum_i \frac{d_i}{2E_t} F_{i \rightarrow H_u}(t), \quad (3)$$

where the sum runs over all the nodes of $G(t)$. Therefore the probability generating functions of GFPT to hub H_u can be expressed as

$$\Phi_{GFPT}(t, z) = \sum_{n=0}^{+\infty} z^n P\{F_{H_u}(t) = n\}. \quad (4)$$

In order to obtain $\Phi_{FRT}(t, z)$ and $\Phi_{GFPT}(t, z)$, we first recall their connections with $\Phi_{RT}(t, z)$, which denotes the probability generating function of the return time for hub H_0 in $G(t)$, and is defined by

$$\Phi_{RT}(t, z) = \sum_{n=0}^{+\infty} z^n P(T_{H_0 \rightarrow H_0}(t) = n), \quad (5)$$

where $T_{H_0 \rightarrow H_0}(t)$ denotes the return time (note: maybe it is not the first time the random walker return to the starting site) on network $G(t)$ for a random walker starting from site H_0 , and $P(T_{H_0 \rightarrow H_0}(t) = n)$ is the probability that a random walker, starting from site H_0 , is found at site H_0 at time n on network $G(t)$.

It is known that⁴⁷,

$$\Phi_{FRT}(t, z) = 1 - 1/\Phi_{RT}(t, z) \quad (6)$$

and

$$\Phi_{GFPT}(t, z) = \frac{z \cdot 2^{t-1}}{(1 - z)w^t} \times \frac{1}{\Phi_{RT}(t, z)}. \quad (7)$$

Therefore, we can calculate $\Phi_{FRT}(t, z)$ and $\Phi_{GFPT}(t, z)$ if $\Phi_{RT}(t, z)$ is obtained.

As derived in Appendix C, for any $t > 0$, $\Phi_{RT}(t, z)$ satisfy the following recurrence relation

$$\Phi_{RT}(t, z) = \Phi_{RT}(t - 1, z)/\psi(\Phi_{FPT}(t - 1, z)), \quad (8)$$

with $\Phi_{RT}(0, z) = \frac{1}{1-z^2}$ and

$$\psi(z) \equiv \frac{\Phi_{RT}(0, z)}{\Phi_{RT}(1, z)}, \quad (9)$$

where the method to calculate $\Phi_{RT}(1, z)$ is presented in Appendix B.

Replacing $\Phi_{RT}(t, z)$ from Eq. (8) in Eqs. (6) and (7), for any $t > 0$, we find that $\Phi_{FRT}(t, z)$ and $\Phi_{GFPT}(t, z)$ satisfy the following recurrence relations,

$$\Phi_{FRT}(t, z) = 1 - \psi(\Phi_{FPT}(t-1, z))[1 - \Phi_{FRT}(t-1, z)], \quad (10)$$

and

$$\Phi_{GFPT}(t, z) = \frac{2}{u+v} \psi(\Phi_{FPT}(t-1, z)) \Phi_{GFPT}(t-1, z), \quad (11)$$

with initial condition $\Phi_{FRT}(0, z) = z^2$, $\Phi_{GFPT}(0, z) = \frac{1}{2}(z+1)z$.

As we have presented method to calculate $\Phi_{FPT}(t, z)$ for any $t > 0$ in Appendix E. Therefore, we can calculate $\Phi_{FRT}(t, z)$ and $\Phi_{GFPT}(t, z)$ by using Eqs. (10) and (11).

IV. EXACT CALCULATION OF THE FIRST AND SECOND MOMENT OF FRT

Calculating the first order derivative with respect to z on both sides of Eq. (10), we find, for any $t > 0$,

$$\begin{aligned} & \frac{\partial}{\partial z} \Phi_{FRT}(t, z) \\ &= \psi(\Phi_{FPT}(t-1, z)) \times \frac{\partial}{\partial z} \Phi_{FRT}(t-1, z) \\ &+ \frac{\partial}{\partial x} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \\ &\times [\Phi_{FRT}(t-1, z) - 1]. \end{aligned} \quad (12)$$

Noting that $\Phi_{FRT}(t, 1) = 1$, $\Phi_{FPT}(t, 1) = 1$ and letting $z = 1$ in Eq. (12), we get, for any $t > 0$,

$$\begin{aligned} \langle FRT_t \rangle &= \frac{\partial}{\partial z} \Phi_{FRT}(t, z) \Big|_{z=1} = a_1 \times \langle FRT_{t-1} \rangle \\ &= \dots = a_1^t \langle FRT_0 \rangle, \end{aligned} \quad (13)$$

where $a_1 \equiv \psi(1)$ and $\langle FRT_t \rangle$ denotes the mean first return time of hub H_0 on $G(t)$. It is well known that⁴², $\langle FRT_t \rangle$ is controlled by the degree of node H_0 , and it can also be expressed as

$$\langle FRT_t \rangle = \frac{2E_t}{d_{H_0}} = 2 \times \left(\frac{u+v}{2} \right)^t. \quad (14)$$

By comparing Eq. (14) with Eq. (13), we get $a_1 = \frac{u+v}{2}$.

Similarity, we can also calculate the second moment of the FRT of hub H_0 , referred to as $\langle FRT_t^2 \rangle$. By taking the first order derivative on both sides of Eq. (12), we

obtain

$$\begin{aligned} & \frac{\partial^2}{\partial z^2} \Phi_{FRT}(t, z) \\ &= \psi(\Phi_{FPT}(t-1, z)) \times \frac{\partial^2}{\partial z^2} \Phi_{FRT}(t-1, z) \\ &+ 2 \frac{\partial}{\partial x} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \\ &\times \frac{\partial}{\partial z} \Phi_{FRT}(t-1, z) + [\Phi_{FRT}(t-1, z) - 1] \\ &\times \frac{\partial^2}{\partial z^2} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z). \end{aligned} \quad (15)$$

By posing $z = 1$ in Eq. (15), we get

$$\begin{aligned} & \frac{\partial^2}{\partial z^2} \Phi_{FRT}(t, z) \Big|_{z=1} \\ &= a_1 \times \frac{\partial^2}{\partial z^2} \Phi_{FRT}(t-1, z) \Big|_{z=1} \\ &+ 2a_2 \times \langle FRT_{t-1} \rangle \times \langle FPT_{t-1} \rangle, \end{aligned} \quad (16)$$

where $a_1 \equiv \psi(1)$, $a_2 \equiv \frac{\partial}{\partial z} \psi(z)|_{z=1}$ and $\langle FPT_t \rangle$ denotes the mean first passage time from hub H_0 to hub H_u on $G(t)$. Therefore, the second moment of FRT of node H_0 satisfies

$$\begin{aligned} \langle FRT_t^2 \rangle &= \frac{\partial^2}{\partial z^2} \Phi_{FRT}(t, z) \Big|_{z=1} + \langle FRT_t \rangle \\ &= a_1 \langle FRT_{t-1}^2 \rangle + 2a_2 \langle FRT_{t-1} \rangle \langle FPT_{t-1} \rangle. \end{aligned} \quad (17)$$

Using Eq. (17) recursively, and inserting Eqs. (13) and (E3) into it, we get

$$\begin{aligned} \langle FRT_t^2 \rangle &= a_1^t \langle FRT_0^2 \rangle + 2a_2 \langle FRT_{t-1} \rangle \langle FPT_{t-1} \rangle \\ &\times \left[1 + \frac{1}{\langle FPT_1 \rangle} + \dots + \frac{1}{\langle FPT_1 \rangle^{t-1}} \right] \\ &= a_1^t \langle FRT_0^2 \rangle + 2 \frac{a_2}{a_1 \langle FPT_1 \rangle} \langle FRT_t \rangle \langle FPT_t \rangle \\ &\times \frac{1 - \frac{1}{\langle FPT_1 \rangle^t}}{1 - \frac{1}{\langle FPT_1 \rangle}}, \end{aligned} \quad (18)$$

which shows that

$$\langle FRT_t^2 \rangle \sim \langle FRT_t \rangle \langle FPT_t \rangle. \quad (19)$$

Therefore the variance of the FRT, denoted by $Var(FRT_t)$, satisfies

$$\begin{aligned} Var(FRT_t) &= \langle FRT_t^2 \rangle - \langle FRT_t \rangle^2 \\ &\sim \langle FRT_t \rangle \langle FPT_t \rangle \times \left[1 - \frac{FRT_t}{FPT_t} \right]. \end{aligned} \quad (20)$$

As shown in Eqs. (E5), (14), and (37), $\langle FPT_t \rangle = (uv)^t$, $\langle FRT_t \rangle = 2 \left(\frac{u+v}{2} \right)^t$ and $\langle GFPT_t \rangle \sim (uv)^t$. Therefore $\langle GFPT_t \rangle \sim \langle FPT_t \rangle$ and $\frac{FRT_t}{FPT_t} \rightarrow 0$ while $t \rightarrow \infty$. Thus, for networks of large size,

$$Var(FRT_t) \sim \langle FRT_t \rangle \langle GFPT_t \rangle. \quad (21)$$

As discussed in Sec.V, $\langle GFPT_t \rangle$ is controlled by the spectral (or transspectral) dimension of the network. Therefore Eqs. (21) shows that $Var(FRT_t)$ is controlled by the degree of node H_0 and the spectral (or transspectral) dimension of the network.

By determining every parameters in Eq. (18), we can further obtain the exactly formula of $\langle FRT_t^2 \rangle$ and $Var(FRT_t)$. Recalling that $\Phi_{FRT}(0, z) = z^2$ and calculating the first and second order derivative of $\Phi_{FRT}(0, z)$ with respect to z and fixing $z = 1$, we obtain $\langle FRT_0 \rangle = 2$ and

$$\langle FRT_0^2 \rangle = 4. \quad (22)$$

For the parameter a_2 , it can be obtained by calculating the first order derivative of $\psi(z)$ with respect to z and fixing $z = 1$. Therefore $\langle FRT_t^2 \rangle$ can be exactly determined by using Eq. (18) and $Var(FRT_t)$ can also be calculated by $Var(FRT_t) = \langle FRT_t^2 \rangle - \langle FRT_t \rangle^2$.

Here we take the specific case of $u = 2$ and $v = 2$ as an example. As derived in Appendix B, $\psi(z) = \frac{2}{2-z^2}$. Thus $a_2 = \frac{\partial}{\partial z} \psi(z)|_{z=1} = 4$ and all parameters in Eq. (18) are known. Therefore $\langle FRT_t^2 \rangle = \frac{8^{t+1}}{3} + \frac{2^{t+2}}{3}$ and

$$Var(FRT_t) = \frac{8^{t+1}}{3} + \frac{2^{t+2}}{3} - 4^{t+1}. \quad (23)$$

In order to evaluate the fluctuation of the FRT, we calculate the reduced moment⁴⁶, defined by $R(FRT_t) = \sqrt{Var(FRT_t)}/\langle FRT_t \rangle$. We find it grows with the increasing of the network and in the limit of large size (i.e., while $t \rightarrow \infty$),

$$R(FRT_t) \sim \left(\frac{2uv}{u+v} \right)^{t/2} \rightarrow \infty, \quad (24)$$

which shows that, on the (u, v) flower of large size, the FRT of hub node has huge fluctuation and the estimate provided by MFRT is unreliable.

V. EXACT CALCULATION OF THE FIRST AND SECOND MOMENT OF THE GFPT

By calculating the first order derivative with respect to z on both sides of Eq. (11), for any $t > 0$, we find,

$$\begin{aligned} & \frac{\partial}{\partial z} \Phi_{GFPT}(t, z) \\ &= \frac{2}{u+v} \left\{ \psi(\Phi_{FPT}(t-1, z)) \times \frac{\partial}{\partial z} \Phi_{GFPT}(t-1, z) \right. \\ &+ \frac{\partial}{\partial x} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \\ &\left. \times \Phi_{GFPT}(t-1, z) \right\}. \end{aligned} \quad (25)$$

Noting that $\Phi_{GFPT}(t, 1) = 1$, $\Phi_{FPT}(t, 1) = 1$, $\psi(1) = \frac{u+v}{2}$ and posing $z = 1$ in Eq. (25), we get, for any $t > 0$,

$$\begin{aligned} \langle GFPT_t \rangle &= \frac{\partial}{\partial z} \Phi_{GFPT}(t, z) \Big|_{z=1} \\ &= \langle GFPT_{t-1} \rangle + \frac{2a_2}{u+v} \langle FPT_{t-1} \rangle, \end{aligned} \quad (26)$$

where a_2 is the same as that of Eq. (16) and $\langle GFPT_t \rangle$ denotes the mean of global first passage time to hub H_u on $G(t)$.

Using Eq. (26) recursively, and replacing $\langle FPT_k \rangle$ from Eq. (E3),

$$\begin{aligned} \langle GFPT_t \rangle &= \langle GFPT_{t-1} \rangle + \frac{2a_2}{u+v} \langle FPT_{t-1} \rangle \\ &= \langle GFPT_{t-2} \rangle + \frac{2a_2}{u+v} \langle FPT_{t-2} \rangle + \frac{2a_2}{u+v} \langle FPT_{t-1} \rangle \\ &= \dots \\ &= \langle GFPT_0 \rangle + \frac{2a_2}{u+v} [\langle FPT_0 \rangle + \langle FPT_1 \rangle + \dots \\ &+ \langle FPT_{t-1} \rangle] \\ &= \langle GFPT_0 \rangle + \frac{2a_2}{u+v} [1 + uv + \dots + (uv)^{t-1}] \\ &= \frac{3}{2} + \frac{2a_2}{u+v} (uv)^{t-1} \frac{1 - (uv)^{-t}}{1 - (uv)^{-1}}, \end{aligned} \quad (27)$$

where $\langle GFPT_0 \rangle = \frac{3}{2}$, which can be easily calculated on the network of generation 0.

Similarity, we can also calculate the second moment of the GFPT to hub H_u , referred to as $\langle GFPT_t^2 \rangle$. By taking the first order derivative on both sides of Eq. (25), we obtain, for any $t > 0$,

$$\begin{aligned} & \frac{\partial^2}{\partial z^2} \Phi_{GFPT}(t, z) \\ &= \frac{2}{u+v} \left\{ \psi(\Phi_{FPT}(t-1, z)) \times \frac{\partial^2}{\partial z^2} \Phi_{GFPT}(t-1, z) \right. \\ &+ 2 \frac{\partial}{\partial x} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \\ &\times \frac{\partial}{\partial z} \Phi_{GFPT}(t-1, z) + \Phi_{GFPT}(t-1, z) \\ &\times \left[\frac{\partial^2}{\partial x^2} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \left(\frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \right)^2 \right. \\ &\left. \left. + \frac{\partial}{\partial x} \psi(x)|_{x=\Phi_{FPT}(t-1, z)} \times \frac{\partial^2}{\partial z^2} \Phi_{FPT}(t-1, z) \right] \right\} \end{aligned} \quad (28)$$

By posing $z = 1$ in Eq. (28), for any $t > 0$, we get

$$\begin{aligned} & \frac{\partial^2}{\partial z^2} \Phi_{GFPT}(t, z) \Big|_{z=1} \\ &= \frac{2}{u+v} \left\{ a_1 \times \frac{\partial^2}{\partial z^2} \Phi_{GFPT}(t-1, z) \Big|_{z=1} \right. \\ &+ 2a_2 \langle GFPT_{t-1} \rangle \langle FPT_{t-1} \rangle + a_3 (\langle FPT_{t-1} \rangle)^2 \\ &\left. + a_2 \times \frac{\partial^2}{\partial z^2} \Phi_{FPT}(t-1, z) \Big|_{z=1} \right\}, \end{aligned} \quad (29)$$

where $a_1 = \frac{u+v}{2}$, a_2 are the same as that of Eq. (16) and

$a_3 \equiv \frac{\partial^2}{\partial z^2} \psi(z)|_{z=1}$. Therefore, for any $t > 0$,

$$\begin{aligned} \langle GFPT_t^2 \rangle &= \frac{\partial^2}{\partial z^2} \Phi_{GFPT}(t, z) \Big|_{z=1} + \langle GFPT_t \rangle \\ &= \langle GFPT_{t-1}^2 \rangle + \frac{4a_2}{u+v} \langle GFPT_{t-1} \rangle \langle FPT_{t-1} \rangle \\ &\quad + \frac{2a_3}{u+v} (\langle FPT_{t-1} \rangle)^2 + \frac{2a_2}{u+v} \langle FPT_{t-1}^2 \rangle, \end{aligned} \quad (30)$$

where $\langle FPT_t^2 \rangle$ denotes the second moment of the first passage time from hub H_0 to hub H_u on $G(t)$.

Replacing $\langle GFPT_{t-1} \rangle$, $\langle FPT_{t-1} \rangle$ and $\langle FPT_{t-1}^2 \rangle$ from Eqs. (27), (E3) and (E8) respectively in Eq. (30), for any $t > 0$,

$$\begin{aligned} \langle GFPT_t^2 \rangle &= \langle GFPT_{t-1}^2 \rangle + \frac{4a_2}{u+v} \langle GFPT_{t-1} \rangle \langle FPT_{t-1} \rangle \\ &\quad + \frac{2a_3}{u+v} (\langle FPT_{t-1} \rangle)^2 + \frac{2a_2}{u+v} \langle FPT_{t-1}^2 \rangle \\ &= \langle GFPT_{t-1}^2 \rangle + k_1 (uv)^{2t-2} + k_2 (uv)^{t-1}, \end{aligned} \quad (31)$$

where

$$k_1 = \frac{8a_2^2}{w^2(uv-1)} + \frac{2a_3}{u+v} + \frac{2a_2(\langle FPT_1^2 \rangle - \langle FPT_1 \rangle)}{wuv(uv-1)}, \quad (32)$$

and

$$k_2 = \frac{8a_2}{u+v} - \frac{8a_2^2}{w^2(uv-1)} - \frac{2a_2(\langle FPT_1^2 \rangle - \langle FPT_1 \rangle)}{wuv(uv-1)}. \quad (33)$$

Using Eq. (31) recursively, for any $t > 0$,

$$\begin{aligned} \langle GFPT_t^2 \rangle &= \langle GFPT_{t-1}^2 \rangle + k_1 (uv)^{2t-2} + k_2 (uv)^{t-1} \\ &= \langle GFPT_{t-2}^2 \rangle + k_1 (uv)^{2t-4} + k_2 (uv)^{t-2} \\ &\quad + k_1 (uv)^{2t-2} + k_2 (uv)^{t-1} \\ &= \dots \\ &= \langle GFPT_0^2 \rangle + k_1 [1 + (uv)^2 + \dots + (uv)^{2t-2}] \\ &\quad + k_2 [1 + uv + \dots + (uv)^{t-1}] \\ &= \langle GFPT_0^2 \rangle + k_1 \frac{(uv)^{2t} - 1}{(uv)^2 - 1} + k_2 \frac{(uv)^t - 1}{uv - 1}. \end{aligned} \quad (34)$$

Because the network of generation 0 is just an edge, it is easy to obtain $\langle GFPT_0^2 \rangle = \frac{5}{2}$. Given u and v , we can derive $\Phi_{RT}(1, z)$ and $\Phi_{FPT}(1, z)$ by using the method presented in Appendix B. Then $\psi(z)$, a_2 , a_3 , $\langle FPT_1 \rangle$ and $\langle FPT_1^2 \rangle$ can all be obtained. Thus k_1 and k_2 is obtained. Therefore $\langle GFPT_t^2 \rangle$ can be exactly calculated and the the variance of the GFPT, denoted by $Var(GFPT_t)$, can also be calculated exactly by $Var(GFPT_t) = \langle GFPT_t^2 \rangle - [\langle GFPT_t \rangle]^2$.

Here we take the specific case of $u = 2$ and $v = 2$ as an example. As derived in Appendix B, $\psi(z) = \frac{2}{2-z^2}$ and $\Phi_{FPT}(1, z) = \frac{-z^2}{z^2-2}$. Thus $a_2 = \frac{\partial}{\partial z} \psi(z)|_{z=1} = 4$, $a_3 = \frac{\partial^2}{\partial z^2} \psi(z)|_{z=1} = 20$, and $\langle FPT_1^2 \rangle - \langle FPT_1 \rangle = \frac{\partial^2}{\partial z^2} \Phi_{FPT}(1, z)|_{z=1} = 20$. Inserting all parameters into Eqs. (32), (33) (27) and (34), we obtain $k_1 = 16$, $k_2 = 2$,

$\langle GFPT_t \rangle = \frac{2}{3} \times 4^t + \frac{5}{6}$, $\langle GFPT_t^2 \rangle = \frac{16^{t+1}}{15} + \frac{2}{3} \times 4^t + \frac{23}{30}$ and

$$Var(GFPT_t) = \frac{28}{45} \times 16^t - \frac{1}{9} \times 4^{t+1} + \frac{13}{180}. \quad (35)$$

We can also find from Eqs. (27) and (34) that $\langle GFPT_t \rangle \sim (uv)^t$ and $\langle GFPT_t^2 \rangle \sim (uv)^{2t}$. Therefore

$$Var(GFPT_t) \sim \langle GFPT_t^2 \rangle \sim (uv)^{2t}. \quad (36)$$

Since the volume of the (u, v) flower scales as $N_t \sim (u+v)^t$ for large sizes,

$$\langle GFPT_t \rangle \sim N_t^{\frac{\ln(uv)}{\ln(u+v)}}, \quad (37)$$

and

$$Var(GFPT_t) \sim N_t^{\frac{2\ln(uv)}{\ln(u+v)}}. \quad (38)$$

For the case of $u > 1$, the (u, v) flowers have spectral dimension $d_s = \frac{2\ln(u+v)}{\ln(uv)}$ ¹³. Therefore Eqs. (37) and (38) show that, $\langle GFPT_t \rangle \sim N_t^{2/d_s}$ and $Var(GFPT_t) \sim N_t^{4/d_s}$. Both the mean and variance of the GFPT are controlled by the spectral dimension d_s .

For the case of $u = 1$, the (u, v) flowers are non-fractal, they have spectral dimension $d_s = 2$ ¹³. Our results show that both the mean and variance of the GFPT have no direct relation with d_s . Although the $(1, v)$ flowers are not fractals, they are transfractals with the transfractal dimension $\tilde{d}_f = \ln(1+v)/(v-1)$ and the transwalk dimension $\tilde{d}_w = \ln(v)/(v-1)$ ¹⁰. Similar to $d_s = 2\tilde{d}_f/\tilde{d}_w$, we define the transspectral dimension by $\tilde{d}_s = 2\tilde{d}_f/\tilde{d}_w = 2\ln(1+v)/\ln(v)$ ⁴⁸, Eqs. (37) and (38) show that, $\langle GFPT_t \rangle \sim N_t^{2/\tilde{d}_s}$ and $Var(GFPT_t) \sim N_t^{4/\tilde{d}_s}$. Therefore the mean and variance of the FRT are controlled by the transspectral dimension \tilde{d}_s .

In order to evaluate the fluctuation of the GFPT, we can further calculate the reduced moment of GFPT⁴⁶, defined by $R(GFPT_t) = \sqrt{Var(GFPT_t)/\langle GFPT_t \rangle}$. Result shows that it grows with the increasing of the network and in the limit of large size (i.e., while $t \rightarrow \infty$),

$$R(GFPT_t) \approx \sqrt{\frac{k_1 w^2 (uv-1)^2}{4a_2^2 (u^2 v^2 - 1)}} - 1. \quad (39)$$

By comparing the result with that of the FRT, the fluctuation of the GFPT is much smaller and the estimate provided by its mean is more reliable.

VI. CONCLUSIONS

We have presented method to calculate exactly the mean and the variance of the GFPT and FRT for a given hub on the (u, v) flowers and the scaling behavior of mean and the variance of the GFPT and FRT were also analyzed. Results show that, for the case of $u > 1$, while the

networks are fractals, both the mean and the variance of the GFPT are controlled by the spectral dimension d_s . However, for the case of $u = 1$, while the networks are nonfractals, the mean and the variance of the GFPT are not controlled by the spectral dimension, but they are controlled by the transspectral dimension. Results also show that the variance of the GFPT and FRT scale with the mean of the GFPT and FRT as $Var(GFPT_t) \sim \langle GFPT_t \rangle^2$ and $Var(FRT_t) \sim \langle FRT_t \rangle \langle GFPT_t \rangle$. Note that the mean of the FRT is controlled by the degree of the node. We found that the variance of the FRT are controlled by the degree of the node and the spectral dimension (or transspectral dimension) of the network.

We have also calculated the reduced moments of the GFPT and FRT and find that, in the limit of large size, the reduced moment of the FRT tends to be infinite, whereas the reduced moment of the GFPT is almost a const. Therefore, on the (u, v) flowers of large size, the FRT has huge fluctuation and the estimate provided by its mean is unreliable, whereas the fluctuation of the GFPT is much smaller and the estimate provided by its mean is more reliable.

Of course, the method proposed here can also be used on other self-similar graph such as Sierpinski gaskets, tree-like fractals, recursive scale-free trees and etc.

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Appendix A: Probability generating function and its properties

Let T be a discrete random variable which takes only non-negative integer values, and whose probability distribution is p_k ($k = 0, 1, 2, \dots$). The probability generating function of T is defined as

$$\Phi_T(z) = \sum_{k=0}^{+\infty} z^k p_k. \quad (A1)$$

The probability generating function of T is determined by the probability distribution and, in turn, it uniquely determines the probability distribution. If T_1 and T_2 are two random variables with the same probability generating function, then they have the same probability distribution. Given the probability generating function $\Phi_T(z)$ of the random variable T , we can obtain the probability distribution p_k ($k = 0, 1, 2, \dots$) as the coefficient of z^k in the Taylor's series expansion of $\Phi_T(z)$ about $z = 0$.

Also, the n -th moment $\langle T^n \rangle \equiv \sum_{k=0}^{+\infty} k^n p_k$, can be written in terms of combinations of derivatives (up to the

n -th order) of $\Phi_T(z)$ calculated in $z = 1$. In particular,

$$\langle T \rangle = \left. \frac{\partial \Phi_T(z)}{\partial z} \right|_{z=1}, \quad (A2)$$

$$\langle T^2 \rangle = \left. \frac{\partial^2 \Phi_T(z)}{\partial z^2} \right|_{z=1} + \left. \frac{\partial \Phi_T(z)}{\partial z} \right|_{z=1}. \quad (A3)$$

Finally, we list some useful properties of the probability generating function⁴⁹, which would be useful in this paper:

- Let T_1 and T_2 be two independent random variables with probability generating functions $\Phi_{T_1}(z)$ and $\Phi_{T_2}(z)$, respectively. Then, the probability generating function of random variable $T_1 + T_2$ reads as

$$\Phi_{T_1+T_2}(z) = \Phi_{T_1}(z)\Phi_{T_2}(z). \quad (A4)$$

- Let N, T_1, T_2, \dots be independent random variables. If T_i ($i = 1, 2, \dots$) are identically distributed, each with probability generating function $\Phi_T(z)$, and, being $\Phi_N(z)$ the probability generating function of N , the random variable defined as

$$S_N = T_1 + T_2 + \dots + T_N \quad (A5)$$

has probability generating function

$$\Phi_{S_N}(z) = \Phi_N(\Phi_T(z)). \quad (A6)$$

Appendix B: Probability generating function of FPT and return time on the networks of generation 1

The structure of (u, v) flower of generation 1 is a ring with $w = u + v$ nodes, which are labeled as $H_0, H_1, H_2, \dots, H_{w-1}$. In this appendix, we present method to calculate $\Phi_{RT}(1, z)$ (i.e., the probability generating function (PGF) of the return time for hub H_0), $\Phi_{FPT}(1, z)$ (i.e., the PGF of the FPT from H_0 to H_u) and $\Phi_{RT}^a(1, z)$ (i.e., the PGF of the return time for hub H_0 in the presence of a trap H_u).

Let

$$M = (P_{ij})_{w \times w}$$

be the transition probability matrix for random walks on the (u, v) flower of generation 1. Here

$$P_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } H_i \sim H_j, \text{ and } x \text{ is not a trap} \\ 0 & \text{others} \end{cases}, \quad (B1)$$

where $H_i \sim H_j$ means that there is an edge between H_i and H_j and d_i is the degree of node H_i . Then we can calculate the probability generating function directly by

$$\Psi(z) = \sum_{n=0}^{+\infty} (zM)^n = (I - zM)^{-1}, \quad (B2)$$

where $\Psi(z) = (\Psi_{ij}(z))_{w \times w}$ and $\Psi_{ij}(z)$ is the PGF of passage time from node H_i to H_j , whereas $\Psi_{ii}(z)$ is the

PGF of return time of node H_i . If H_j is a trap, $\Psi_{ij}(z)$ is just the PGF of first passage time from node H_i to H_j and $\Psi_{ii}(z)$ is just the PGF of the return time in the presence of a trap H_j for random walkers starting from node H_i .

Exact calculation of $\Phi_{RT}(1, z)$ and $\psi(z)$ on (2, 2) flower and (1, 3) flower. For network of generation 1, the structure of (2, 2) flower and (1, 3) flower are the same. Therefore $\Phi_{RT}(1, z)$ is the equal for (2, 2) flower and (1, 3) flower. In order to calculate $\Phi_{RT}(1, z)$, no trap is introduced in the network. Therefore,

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}. \quad (\text{B3})$$

Replacing M from Eq. (B3) in Eq. (B2), calculating $(I - zM)^{-1}$ by using the tools of *Matlab*, we obtain $\Psi(z)$ for this case. Then the probability generating function of the RT of hub H_0 is

$$\Phi_{RT}(1, z) = \Psi_{00}(z) = \frac{z^2 - 2}{2(z^2 - 1)}. \quad (\text{B4})$$

Notice that $\Phi_{RT}(0, z) = \frac{1}{1 - z^2}$, we obtain

$$\psi(z) \equiv \frac{\Phi_{RT}(0, z)}{\Phi_{RT}(1, z)} = \frac{2}{2 - z^2}. \quad (\text{B5})$$

Exact calculation of $\Phi_{FPT}(1, z)$ and $\Phi_{RT}^a(1, z)$ on (2, 2) flower and (1, 3) flower. Firstly we calculate $\Phi_{FPT}(1, z)$, $\Phi_{RT}^a(1, z)$ on (2, 2) flower. Let hub H_2 be a trap. Therefore,

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}. \quad (\text{B6})$$

Calculating $\Psi(z)$ from Eq. (B2) by using *Matlab*, the probability generating function of the FPT from H_0 to H_u on (2, 2) flower is

$$\Phi_{FPT}(1, z) = \Psi_{02}(z) = \frac{-z^2}{z^2 - 2}, \quad (\text{B7})$$

and the probability generating function of the return time of hub H_0 in the presence of a trap H_2 on (2, 2) flower reads as

$$\Phi_{RT}^a(1, z) = \Psi_{00}(z) = \frac{-2}{z^2 - 2}. \quad (\text{B8})$$

Furthermore, we can also derive $\Phi_{FPT}(1, z)$, $\Phi_{RT}^a(1, z)$ for (1, 3) flower from $\Psi(z)$ we just get. The results are

$$\Phi_{FPT}(1, z) = \Psi_{12}(z) = \frac{-z}{z^2 - 2}, \quad (\text{B9})$$

and

$$\Phi_{RT}^a(1, z) = \Psi_{11}(z) = \frac{z^2 - 4}{2(z^2 - 2)}. \quad (\text{B10})$$

Note: the structure of (1, 3) flower of generation 1 is a ring with 4 nodes, which is the same as that of (2, 2) flower. The first passage time from H_0 to H_3 is the same as the first passage time from H_1 to H_2 and the return time of hub H_0 in the presence of a trap H_3 is the same as the return time of hub H_1 in the presence of a trap H_2 .

Appendix C: Derivation of Eq. (8)

Considering any return path π starting from H_0 and ending at H_0 on $G(t)$. Its length is just the return time $T_{H_0 \rightarrow H_0}(t)$. Let v_i be the node of $G(t)$ reached at time i . Then the path can be rewritten as

$$\pi = (v_0 = H_0, v_1, v_2, \dots, v_{T_{H_0 \rightarrow H_0}(t)} = H_0).$$

In general, H_0 and H_u would appear in the path π for many times. We denote with Ω the set of nodes $\{H_0, H_u\}$ and introduce the observable $\tau_i = \tau_i(\pi)$, which is defined recursively as follows:

$$\tau_0(\pi) = 0, \quad (\text{C1})$$

$$\tau_i(\pi) = \min\{k : k > \tau_{i-1}, v_{\tau_i} \in \Omega, v_{\tau_i} \neq v_{\tau_{i-1}}\}. \quad (\text{C2})$$

Then, considering only nodes in the set Ω , the path π can be restated into a “simplified path” defined as

$$\iota(\pi) = (v_{\tau_0} = H_0, v_{\tau_1}, \dots, v_{\tau_L} = H_0), \quad (\text{C3})$$

where $L = \max\{i : v_{\tau_i} = H_0\}$, which is the total number of observables obtained from the path π . In fact, the “simplified path” is obtained by removing any nodes of π except H_0 and H_u . If node H_0 (or H_u) appears consecutively in the “simplified path”, we just record the first one.

Note that Ω represents the nodes set of the (u,v) flower with generation 0 and the path $\iota(\pi)$ includes only the nodes of Ω . Thus, $\iota(\pi)$ is just a return path of H_0 on the (u,v) flower with generation 0 and L is just the return time of H_0 on the (u,v) flower with generation 0. Therefore, the probability generating function of L is $\Phi_{RT}(0, z)$.

For any return path π of H_0 , maybe v_{τ_L} is not the last node of path π . That is to say, after node v_{τ_L} , path π includes a sub-path from H_0 to H_0 , which does not reach H_u . In principle, the sub-path may include any node of $G(t)$ except H_u . Therefore, the sub-path can be regarded as a return path of H_0 in the presence of an absorbing hub H_u . We denote its length by $T_{H_0 \rightarrow H_0}^a(t)$ and denote its probability generating function by $\Phi_{RT}^a(t, z)$.

Let T_i ($i = 1, 2, \dots, L$) be the time taken to move from $v_{\tau_{i-1}}$ to v_{τ_i} , namely $T_i = \tau_i - \tau_{i-1}$. Therefore the return

time $T_{H_0 \rightarrow H_0}(t)$ on $G(t)$ satisfies

$$\begin{aligned} T_{H_0 \rightarrow H_0}(t) &= \tau_L - \tau_0 + T_{C \rightarrow C}^a(t-1) \\ &= T_1 + T_2 + \cdots + T_L + T_{H_0 \rightarrow H_0}^a(t). \end{aligned} \quad (C4)$$

Because $v_{\tau_i} = H_0$ (or H_u) for any $i = 0, 1, 2, \dots, L$ and $v_{\tau_i} \neq v_{\tau_{i-1}}$ for any $i > 0$. Then T_i ($i = 1, 2, \dots, L$) are identically distributed random variables, each of them is the first-passage time from hub H_0 to hub H_u (or from hub H_u to hub H_0). Its probability generating function is denoted by $\Phi_{FPT}(t, z)$.

Note that L , $T_{H_0 \rightarrow H_0}^a(t)$, T_1 , T_2 , \dots are independent random variables. According to the properties (see Eqs. (A4) and (A6)) of the probability generating function presented in Appendix A, the probability generating function of return time $T_{H_0 \rightarrow H_0}(t)$ satisfies

$$\Phi_{RT}(t, z) = \Phi_{RT}(0, x)|_{x=\Phi_{FPT}(t, z)} * \Phi_{RT}^a(t, z). \quad (C5)$$

As derived in Appendix D, for any $t > 0$,

$$\Phi_{RT}^a(t, z) = \Phi_{RT}^a(1, \Phi_{FPT}(t-1, z)) * \Phi_{RT}^a(t-1, z). \quad (C6)$$

Replacing $\Phi_{RT}^a(t, z)$ from Eq. (C6) in Eq. (C5), for any $t > 0$,

$$\begin{aligned} \Phi_{RT}(t, z) &= \Phi_{RT}(0, \Phi_{FPT}(t, z)) * \Phi_{RT}^a(1, \Phi_{FPT}(t-1, z)) * \Phi_{RT}^a(t-1, z) \\ &= \frac{\Phi_{RT}(0, \Phi_{FPT}(t, z)) * \Phi_{RT}^a(1, \Phi_{FPT}(t-1, z))}{\Phi_{RT}(0, \Phi_{FPT}(t-1, z))} \\ &\quad * \Phi_{RT}(0, \Phi_{FPT}(t-1, z)) * \Phi_{RT}^a(t-1, z) \\ &= \frac{\Phi_{RT}(0, \Phi_{FPT}(t, z)) * \Phi_{RT}^a(1, \Phi_{FPT}(t-1, z))}{\Phi_{RT}(0, \Phi_{FPT}(t-1, z))} \\ &\quad * \Phi_{RT}(t-1, z). \end{aligned}$$

As derived in Appendix E,

$$\Phi_{FPT}(t, z) = \Phi_{FPT}(1, \Phi_{FPT}(t-1, z)).$$

Therefore,

$$\begin{aligned} \Phi_{RT}(0, \Phi_{FPT}(t, z)) &= \Phi_{RT}(0, \Phi_{FPT}(1, \Phi_{FPT}(t-1, z))) \\ &= \Phi_{RT}(0, \Phi_{FPT}(1, x))|_{x=\Phi_{FPT}(t-1, z)}. \end{aligned} \quad (C8)$$

Thus

$$\begin{aligned} \Phi_{RT}(0, \Phi_{FPT}(t, z)) * \Phi_{RT}^a(1, \Phi_{FPT}(t-1, z)) &= [\Phi_{RT}(0, \Phi_{FPT}(1, x)) * \Phi_{RT}^a(1, x)]|_{x=\Phi_{FPT}(t-1, z)} \\ &= \Phi_{RT}(1, x)|_{x=\Phi_{FPT}(t-1, z)}. \end{aligned} \quad (C9)$$

Let

$$\psi(z) \equiv \frac{\Phi_{RT}(0, z)}{\Phi_{RT}(1, z)}.$$

We have

$$\begin{aligned} \frac{\Phi_{RT}(0, \Phi_{FPT}(t, z)) * \Phi_{RT}^a(0, \Phi_{FPT}(t-1, z))}{\Phi_{RT}(0, \Phi_{FPT}(t-1, z))} &= \frac{1}{\psi(x)|_{x=\Phi_{FPT}(t-1, z)}}. \end{aligned} \quad (C10)$$

Inserting it into Eq. (C7), we obtain Eq. (8).

Because the network of generation 0 is just two nodes connected by an edge, the return probability of return in odd times is 0 and the return probability of return in even times is 1. Therefore

$$\Phi_{RT}(0, z) = \sum_{n=0}^{+\infty} z^{2n} = \frac{1}{1-z^2}. \quad (C11)$$

Appendix D: Derivation of Eq. (C6)

Considering any return path π of H_0 on $G(t)$ in the presence of an absorbing hub H_u . It can be written as

$$\pi = (v_0 = H_0, v_1, v_2, \dots, v_{T_{H_0 \rightarrow H_0}^a(t)} = H_0),$$

where v_i is the node reached at time i and $T_{H_0 \rightarrow H_0}^a(t)$ is the length of the path π . Let $\Omega \equiv \{H_0, H_1, \dots, H_{w-1}\}$. Similar to Appendix C, we introduce the observable $\tau_i = \tau_i(\pi)$, which is defined recursively as Eqs. (C1) and (C2). Then, the path π can be restated as a “simplified path” defined as

$$\iota(\pi) = (v_{\tau_0} = H_0, v_{\tau_1}, \dots, v_{\tau_L} = H_0), \quad (D1)$$

where $v_{\tau_0} \in \Omega$ and L is the total number of observables obtained from the path π .

For any return path π of H_0 in the presence of an absorbing hub H_u , similar to the discussion in Appendix C, path π includes a sub-path from H_0 to H_0 after node v_{τ_L} . The sub-path does not reach any node in Ω except for H_0 . Therefore, the sub-path can be regarded as a return path of H_0 on $G(t)$ in the presence of $w-1$ absorbing hubs (i.e., H_1, H_2, \dots, H_{w-1}). In fact, the sub-path only includes nodes of Γ_1 and Γ_w (see Fig. 4), which are copies of $G(t-1)$. By symmetry, nodes of Γ_1 are in one to one corresponding with nodes of Γ_w . If we replaced all the nodes of Γ_w with the corresponding nodes of Γ_1 in the sub-path, we obtain a return path of H_0 in Γ_1 which never reaches hub H_1 . It is a return path of H_0 on Γ_1 in the presence of an absorbing hub H_1 and has the same path length as the original sub-path. If we look Γ_1 as a copy of $G(t-1)$, hub H_1 of $G(t)$ can also be looked as H_u of $G(t-1)$. Therefore the length of the sub-path after node v_{τ_L} can be regarded as the return time of H_0 on $G(t-1)$ in the presence of an absorbing hub H_u .

Let $T_i = \tau_i - \tau_{i-1}$ for any $i = 1, 2, \dots, L$. Therefore the length of the path π satisfies

$$T_{H_0 \rightarrow H_0}^a(t) = T_1 + T_2 + \cdots + T_L + T_{H_0 \rightarrow H_0}^a(t-1). \quad (D2)$$

Note that Ω just represents the nodes set of the (u,v) flower with generation 1 and the path $\iota(\pi)$ includes only the nodes of Ω . Thus, $\iota(\pi)$ is just a return path of H_0 on $G(1)$ in the presence of an absorbing hub H_u and L is just the path length. Therefore, the probability generating function of L is $\Phi_{RT}^a(1, z)$.

Similarly, we find that T_i ($i = 1, 2, \dots$) are identically distributed random variables, each of them can be regarded as the first-passage time from hub H_0 to H_u on $G(t-1)$. Therefore the probability generating function of T_i ($i = 1, 2, \dots$) is $\Phi_{FPT}(t-1, z)$. Thus, we can obtain Eq. (C6) from Eqs. (A4), (A6) and (D2).

Appendix E: Probability generating function of FPT from H_0 to H_u

For the (u, v) flower of generation $t = 1$, the probability generating function of FPT from H_0 to H_u is presented in Eq. (B9). For the (u, v) flower of generation $t > 1$, let $T_{H_0 \rightarrow H_u}(t)$ denotes the FPT H_0 to H_u . Similar to the derivation of Eq. (D2), we can find independent random variables L, T_1, T_2, \dots , such that

$$T_{H_0 \rightarrow H_u}(t) = T_1 + T_2 + \dots + T_L. \quad (\text{E1})$$

Here L is the first-passage time from hub H_0 to H_u on $G(1)$ and T_i ($i = 1, 2, \dots$) are identically distributed random variables, each of them is the first-passage time from hub H_0 to H_u on $G(t-1)$. Therefore the probability generating function of L is $\Phi_{FPT}(1, z)$ and the probability generating function of T_i ($i = 1, 2, \dots$) is $\Phi_{FPT}(t-1, z)$. Thus, we can obtain from Eqs. (A4), (A6) and (E1) that, for any $t > 1$,

$$\Phi_{FPT}(t, z) = \Phi_{FPT}(1, \Phi_{FPT}(t-1, z)). \quad (\text{E2})$$

By taking the first order derivative on both sides of Eq. (E2) and posing $z = 1$, we obtain the mean FPT from H_0 to H_u , i.e., for any $t \geq 1$,

$$\begin{aligned} \langle FPT_t \rangle &= \left. \frac{\partial}{\partial z} \Phi_{FPT}(t, z) \right|_{z=1} \\ &= \left. \frac{\partial}{\partial z} \Phi_{FPT}(1, z) \right|_{z=1} \times \left. \frac{\partial}{\partial z} \Phi_{FPT}(t-1, z) \right|_{z=1} \\ &= \langle FPT_1 \rangle \langle FPT_{t-1} \rangle \\ &= \dots = \langle FPT_1 \rangle^t. \end{aligned} \quad (\text{E3})$$

For $t = 1$, the (u, v) flower $G(1)$ is a ring with w nodes and w links. If we view the networks $G(1)$ as electrical networks by considering each edge to be a unit resistor, the effective resistance between two nodes H_0 and H_u is $R_{H_0 \leftrightarrow H_u} = \frac{uv}{u+v}$. Therefore, the mean FPT from H_0 to H_u is⁵⁰

$$\langle FPT_1 \rangle = E_0 R_{H_0 \leftrightarrow H_u} = uv. \quad (\text{E4})$$

Inserting Eq. (E4) into Eq. (E3), for any $t \geq 1$,

$$\langle FPT_t \rangle = (uv)^t. \quad (\text{E5})$$

It is easy to verify that Eq. (E5) also hold for $t = 0$. Since the volume of the underlying structure scales as $N_t \sim (u+v)^t$ for large sizes, the previous expression shows

$$\langle FPT_t \rangle \sim N_t^{\frac{\ln(uv)}{\ln(u+v)}}. \quad (\text{E6})$$

Similarity, we can also obtain the second moment of the FPT from H_0 to H_u , referred to as $\langle FPT_t^2 \rangle$. Let $\Theta_t \equiv \left. \frac{\partial^2}{\partial z^2} \Phi_{FPT}(t, z) \right|_{z=1}$. By taking the second order derivative on both sides of Eq. (E2) and posing $z = 1$, for any $t \geq 1$,

$$\begin{aligned} \Theta_t &= \langle FPT_1 \rangle \Theta_{t-1} + \Theta_1 (\langle FPT_{t-1} \rangle)^2 \\ &= uv \Theta_{t-1} + \Theta_1 (uv)^{2t-2} \\ &= (uv)^2 \Theta_{t-2} + \Theta_1 (uv)^{2t-3} + \Theta_1 (uv)^{2t-2} \\ &= \dots \\ &= (uv)^{t-1} \Theta_1 + \Theta_1 [(uv)^t + (uv)^{t+1} + \dots + \Theta_1 (uv)^{2t-2}] \\ &= \Theta_1 (uv)^{t-1} \frac{(uv)^t - 1}{uv - 1}. \end{aligned} \quad (\text{E7})$$

For Θ_1 , it can be calculated directed by calculating the second order derivative of $\Phi_{FPT}(1, z)$ and the method for calculating $\Phi_{FPT}(1, z)$ is present in Appendix A. Therefore, for any $t \geq 1$,

$$\begin{aligned} \langle FPT_t^2 \rangle &= \Theta_t + \langle FPT_t \rangle \\ &= \Theta_1 (uv)^{t-1} \frac{(uv)^t - 1}{uv - 1} + (uv)^t, \end{aligned} \quad (\text{E8})$$

where Θ_1 can also be rewritten as $\langle FPT_1^2 \rangle - \langle FPT_1 \rangle$.

It is easy to obtain $\langle FPT_0^2 \rangle = 1$. Therefore Eq. (E8) also holds for $t = 0$.

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